ON THE MULTIPLE ZEROS OF A REAL ANALYTIC FUNCTION WITH APPLICATIONS TO THE AVERAGING THEORY OF DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work we consider real analytic functions $d(z, \lambda, \varepsilon)$, where $d : \Omega \times \mathbb{R}^p \times I \to \Omega$, Ω is a bounded open subset of $\mathbb{R}$, $I \subset \mathbb{R}$ is an interval containing the origin, $\lambda \in \mathbb{R}^p$ are parameters, and ε is a small parameter. We study the branching of the zero-set of $d(z, \lambda, \varepsilon)$ at multiple points when the parameter ε varies.

We apply the obtained results to improve the classical averaging theory for computing T-periodic solutions of λ-families of analytic T-periodic ordinary differential equations defined on $\mathbb{R}$, using the displacement functions $d(z, \lambda, \varepsilon)$ defined by these equations.

We call the coefficients in the Taylor expansion of $d(z, \lambda, \varepsilon)$ in powers of ε the averaged functions. The main contribution consists in analyzing the role that have the multiple zeros $z_0 \in \Omega$ of the first non-zero averaged function. The outcome is that these multiple zeros can be of two different classes depending on whether the zeros (z_0, λ) belong or not to the analytic set defined by the real variety associated to the ideal generated by the averaged functions in the Noetheriang ring of all the real analytic functions at (z_0, λ). We bound the maximum number of branches of isolated zeros that can bifurcate from each multiple zero z_0. Sometimes these bounds depend on the cardinalities of minimal bases of the former ideal. Several examples illustrate our results and they are compared with the classical theory, branching theory and also under the light of singularity theory of smooth maps. The examples range from polynomial vector fields to Abel differential equations and perturbed linear centers.

1. Introduction and statement of the results

The method of averaging is a classical tool that allows to study the dynamics of the periodic nonlinear differential systems. It has a long history starting with the intuitive classical works of Lagrange and Laplace. Important advances of the averaging theory were made by Bogoliubov and Krylov, the reader can consult [2] for example. For a more modern exposition of the averaging theory see the book of Sanders, Verhulst and Murdock [13].

In this work we consider a family of T-periodic analytic differential equations in $\Omega \subset \mathbb{R}$ of the form

$$\dot{x} = F(t, x; \lambda, \varepsilon) = \sum_{i \geq 1} F_i(t, x; \lambda)\, \varepsilon^i, \tag{1}$$

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where t is the independent variable (here called the *time*), and $x \in \Omega$ is the dependent variable with Ω a bounded open subset of $\mathbb{R}$, $\lambda \in \mathbb{R}^p$ are the parameters of the family, for all i the function F_i is analytic in its variables and T-periodic in the t variable, and the period T is independent of the small parameter $\varepsilon \in I$ with $I \subset \mathbb{R}$ an interval containing the origin.

For each $z \in \Omega$ we denote by $x(t; z, \lambda, \varepsilon)$ the solution of the Cauchy problem formed by the differential equation (1) with the initial condition $x(0; z, \lambda, \varepsilon) = z$. From the analyticity of equation (1) and the fact that $F(t, x; \lambda, 0) = 0$ one has

$$x(t; z, \lambda, \varepsilon) = z + \sum_{j \geq 1} x_j(t, z, \lambda)\, \varepsilon^j, \tag{2}$$

where $x_j(t, z, \lambda)$ are real analytic functions such that $x_j(0, z, \lambda) = 0$. Assuming that $x(t; z, \lambda, \varepsilon)$ is defined in the interval $t \in [0, T]$ (this is guarantee for ε close enough to 0, due to the existence and uniqueness of the solutions for the Cauchy problem on the time-scale $1/\varepsilon$), we can define the analytic *displacement map at time* T as $d : \Omega \times \mathbb{R}^p \times I \to \Omega$ with

$$d(z, \lambda, \varepsilon) = x(T; z, \lambda, \varepsilon) - x(0; z, \lambda, \varepsilon) = x(T; z, \lambda, \varepsilon) - z.$$

Clearly, its zeros are initial conditions for the T-periodic solutions of the differential equation (1).

Integrating with respect to the time t the differential equation (1) along the solution $x(t; z, \lambda, \varepsilon)$ from 0 to t we obtain

$$x(t; z, \lambda, \varepsilon) - z = \int_0^t F(s, x(s; z, \varepsilon), \lambda, \varepsilon)\, ds,$$

from which we get

$$d(z, \lambda, \varepsilon) = \int_0^T F(t, x(t; z, \varepsilon), \lambda, \varepsilon)\, dt.$$

The displacement map is analytic at $\varepsilon = 0$, so we can express it as the following series expansion

$$d(z, \lambda, \varepsilon) = \sum_{i \geq 1} f_i(z; \lambda)\, \varepsilon^i. \tag{3}$$

The coefficient functions $f_i(z; \lambda)$ are called the *averaged functions*. The way in which we can get (3) is explained with details in [6]. There we can see that the first coefficient is

$$f_1(z; \lambda) = \int_0^T F_1(t, z, \lambda)\, dt.$$

For the expression of all the other coefficients, see [6] again where the recursive expression of $x_i(t; z, \lambda)$ for $i \geq 1$ is given. We summarize these results in Theorem 22 of the Appendix and we will use it for the computation of the averaged functions taking into account that from (2) we have $f_i(z; \lambda) = x_i(T; z, \lambda)$ for all positive integer i.

We say that a (*complete, or positive, or negative*) *branch of T-periodic solutions of equation* (1) *bifurcates* from the point $z_0 \in \Omega$ if there is continuous function $z^*(\lambda, \varepsilon)$ (defined either for all ε in a neighborhood of zero, or in a half-neighborhood of zero just for all $\varepsilon > 0$ close to zero, or $\varepsilon < 0$ close to zero, respectively) such that $z^*(\lambda, 0) = z_0$ and $d(z^*(\lambda,\varepsilon),\lambda,\varepsilon) \equiv 0$.

Therefore the solutions $x(t; z^*(\lambda,\varepsilon),\lambda,\varepsilon)$ of equation (1) are T-periodic and bifurcate from z_0 because $\lim_{\varepsilon\to 0^\pm} x(t; z^*(\lambda,\varepsilon),\lambda,\varepsilon) = z_0$, where the lateral limit is taken according with the complete, or positive, or negative nature of the branch.

Now we state the following easy result and, for completeness, we prove it in subsection §5.1.

Lemma 1. *Let $z^*(\lambda,\varepsilon)$ be a function defined for all ε in a sufficiently small half-neighborhood of zero such that $d(z^*(\lambda,\varepsilon),\lambda,\varepsilon) \equiv 0$. Then $f_\ell(z_0(\lambda);\lambda) = 0$ where $z^*(\lambda,0) = z_0(\lambda) \in \Omega$ being ℓ the first subindex such that $f_\ell(z;\lambda) \not\equiv 0$.*

By Lemma 1 in order to control the bifurcation of the families of T-periodic solutions of the differential equation (1) for small values of $|\varepsilon|$, we need to study the zeros of the function $f_\ell(.;\lambda)$ defined in Lemma 1.

Given a particular differential equation (1) with $\lambda = \lambda^*$, in the following let ℓ be the first positive integer such that $f_\ell(z;\lambda^*) \not\equiv 0$. Recall that a zero $z_0 \in \Omega$ of $f_\ell(.;\lambda^*)$ is called *multiple* or *simple* according to whether $\dfrac{\partial}{\partial z} f_\ell(z_0;\lambda^*)$ is zero or not, respectively. In the multiple case z_0 has *multiplicity* $\bar{k}$ if this is the minimum integer such that $\dfrac{\partial^{\bar{k}} f_\ell}{\partial z^{\bar{k}}}(z_0;\lambda^*) \neq 0$.

We say that a T-periodic solution $x(t; z,\lambda,\varepsilon)$ is *isolated* if there is a neighborhood $N \subset \Omega$ of z such that $x(t;\hat{z},\lambda,\varepsilon)$ is not T-periodic for all $\hat{z} \in N\backslash\{z\}$.

The next result will be called here the classical averaging theory.

Theorem 2. *Assume that $z_0 \in \Omega$ is a zero of $f_\ell(.;\lambda^*)$. Let N be the number of isolated branches of T-periodic solutions bifurcating from z_0 for equation* (1) *with $\lambda = \lambda^*$ and $|\varepsilon| \ll 1$. Then the following statements hold.*

(i) *If z_0 is simple then $N = 1$ and the branch is complete and analytic.*
(ii) *Assume that z_0 is multiple of multiplicity $\bar{k}$. Then $N \le \bar{k}$. Additionally, if $\bar{k}$ is odd then $N \ge 1$ and is also odd.*

Theorem 2(i) for simple zeros is well known (see for instance [10]) and it is consequence of the Implicit Function Theorem, for more details see §5.2. The first part of Theorem 2(ii) for multiple points can be proved by using several times the Rolle theorem (see [11]), while the last part is consequence of the fact that univariate real polynomials of odd degree always have an odd number (greater or equal than 1) of real roots. Theorem 2 and some generalizations can be found for example in [17].

In the rest of the work when analyzing the role that multiple zeros $z_0 \in \Omega$ of $f_\ell(.;\lambda^*)$ have in these bifurcations, we change the classical strategy of finding the branches of T-periodic solutions of equation (1) bifurcating from $z_0 \in \Omega$ computing the continuous functions $z^*(\lambda,\varepsilon)$ satisfying $z^*(\lambda^*,0) = z_0$ and $d(z^*(\lambda^*,\varepsilon),\lambda^*,\varepsilon) \equiv 0$ for any ε in a half-neighborhood of zero; for finding the continuous functions $\varepsilon^*(z,\lambda^*)$ with $\varepsilon^*(z_0,\lambda^*) = 0$ such that $d(z,\lambda^*,\varepsilon^*(z,\lambda^*)) \equiv 0$ for any z in both half-neighborhoods of z_0. This will be our approach which, in some cases, improves the bound $\bar{k}$ provided in Theorem 2(ii) for multiple zeros.

When using the above strategy we note that in order to associate only one branch $z^*(\lambda^*, \varepsilon)$ to each continuous function $\varepsilon^*(z, \lambda^*)$ it is needed the additional hypothesis that the function $\varepsilon^*(., \lambda^*)$ be *locally invertible*, so that $z^*(\lambda^*, .)$ is the (local) inverse function of $\varepsilon^*(., \lambda^*)$. Notice that, remarkably, it is possible that a nonconstant continuous function of a real variable be not locally invertible in a neighborhood of a point even when that point is not an extreme value of the function. A typical example is the Weierstrass function which is continuous and bounded but nowhere monotone.

In this direction our first result (see the proof in §5.3) is the following theorem. There we show that the former behavior of some continuous functions is not allowed for the branches.

Theorem 3. *The branches $z^*(\lambda, \varepsilon)$ of T-periodic solutions of equation* (1) *bifurcating from the point $z_0 \in \Omega$ are locally invertible near $(z, \varepsilon) = (z_0, 0)$.*

Remark 4. In order to analyze the local structure of the zeroes of the displacement map around $(z, \varepsilon) = (z_0, 0)$, besides branching theory based on Newton's diagram (see [17] and the proof of Theorem 3), another different approach comes from the singularity theory of smooth functions (see for example [7]) applied to the reduced displacement map $\Delta(z, \lambda, \varepsilon) = f_\ell(z; \lambda) + \sum_{i \geq 1} f_{\ell+i}(z; \lambda)\varepsilon^i$. In this approach the goal is to find, for some fixed $\lambda = \lambda^*$ so we remove its dependence, a normal form $\hat{\Delta}(z, \varepsilon)$ of $\Delta(z, \varepsilon)$ which are related by $U(z, \varepsilon)\, \Delta(Z(z, \varepsilon), \Lambda(\varepsilon)) = \hat{\Delta}(z, \varepsilon)$ where $(z, \varepsilon) \mapsto (Z(z, \varepsilon), \Lambda(\varepsilon))$ is a local diffeomorphism of $\mathbb{R}^2$ mapping the origin to $(z_0, 0)$, preserving orientation (i.e. the derivatives are $Z_z(z, \varepsilon) > 0$ and $\Lambda_\varepsilon(\varepsilon) > 0$), and $U(z, \varepsilon)$ is a positive function. Notice that if $N_\Delta(\varepsilon)$ denotes the number of local zeros of $\Delta(., \varepsilon)$ near z_0 then one has the important consequence that $N_\Delta(\varepsilon) = N_{\hat{\Delta}}(\Lambda(\varepsilon))$.

The proof of the forthcoming Theorem 5 is rather similar to that of Theorem 2(i) by using the Weierstrass preparation theorem instead of the Implicit Function Theorem and interchanging the role of z and ε, see the proof in subsection §5.4.

Theorem 5. *For a fixed $\lambda^\dagger \in \mathbb{R}^p$ assume that ℓ is the first subindex of the displacement function* (3) *such that $f_\ell \not\equiv 0$. Let $z_0 \in \Omega$ be a multiple zero of the function $f_\ell(.; \lambda^\dagger)$. Assume also that there exists a positive integer k which is the minimum integer satisfying $f_{\ell+k}(z_0; \lambda^\dagger) \neq 0$. Then for $|\varepsilon|$ sufficiently small the number of (either positive or negative) isolated branches of the T-periodic solutions that equation* (1) *with $\lambda = \lambda^\dagger$ can have bifurcating from z_0 is bounded by $2k$. Moreover, if k is odd then the number of such branches is also odd and at least one branch bifurcates from z_0.*

We note that the upper bound $2k$, obtained in Theorem 5 for the maximum number of isolated branches of T-periodic solutions that equation (1) with $\lambda = \lambda^\dagger$ can have bifurcating from the multiple zero z_0, is not related with the multiplicity $\bar{k}$ of the zero z_0 as it is explained in Theorem 2(ii). In this way we are providing another mechanism to obtain such a maximum bound.

Remark 6. We note that the positive integer k of Theorem 5 may not exist. A typical example is a zero $(z_0, \lambda^*) \in \Omega \times \mathbb{R}^p$ of the displacement function (3) such that z_0 is an equilibrium point of the differential equation (1) with $\lambda = \lambda^*$, i.e. $F(t, z_0; \lambda^*, \varepsilon) = 0$ for all $t \in \mathbb{R}$ and $|\varepsilon| \ll 1$. For such a zero one has $f_i(z_0; \lambda^*) = 0$ for all positive integer i although $f_\ell \not\equiv 0$ for some ℓ.

Based on Remark 6 we need to develop a procedure taking also into account these kind of zeros, and to do a complementary theory for studying them.

1.1. **Multiple zeros of finite-type and of infinite-type.** Assume that $f_1(z; \lambda) = \cdots = f_{\ell-1}(z; \lambda) \equiv 0$ and $f_\ell(z; \lambda) \not\equiv 0$ for some index $\ell \geq 1$, that is, the displacement map of family (1) is given by $d(z, \lambda, \varepsilon) = \sum_{i \geq \ell} f_i(z; \lambda)\varepsilon^i$.

We say that a point $(z, \lambda) = (z_0, \lambda^\dagger) \in \Omega \times \mathbb{R}^p$ is of *finite-type* if there exists an integer $k \geq 1$ such that $f_\ell(z_0; \lambda^\dagger) = \cdots = f_{\ell+k-1}(z_0; \lambda^\dagger) = 0$ but $f_{\ell+k}(z_0; \lambda^\dagger) \neq 0$. We call k the *order* of the zero $(z_0, \lambda^\dagger)$. For example, the point $(z_0, \lambda^\dagger)$ in Theorem 5 is of finite-type. We want to emphasize that the branching analysis performed in the book [17] only applies to points of finite-type, so that the initial point in the set $\mathbb{N}^2$ of the Newton's diagram is $(0, k)$ where k is the order, and the terminal point is $(\bar{k}, 0)$ being $\bar{k}$ the multiplicity of z_0 as root of $f_\ell(.; \lambda^\dagger)$. Of course here $\mathbb{N}$ denotes the set of all positive integers.

We say that a zero $(z_0, \lambda^*) \in \Omega \times \mathbb{R}^p$ of the function f_ℓ is of *infinite-type* when $f_j(z_0; \lambda^*) = 0$ for all positive integer j. Let $\mathbb{R}\{z, \lambda\}_{(z_0,\lambda^*)}$ be the Noetherian ring formed by all the real analytic functions at (z_0, λ^*). It is clear that for (z, λ) sufficiently close to (z_0, λ^*), the sequence $\{f_j(z; \lambda)\}_{j \in \mathbb{N}} \subset \mathbb{R}\{z, \lambda\}_{(z_0,\lambda^*)}$, and we define the ideal $\mathcal{I} = \langle f_i(z; \lambda) : i \in \mathbb{N} \rangle$ in the ring $\mathbb{R}\{z, \lambda\}_{(z_0,\lambda^*)}$ as the ideal generated by all the functions $f_i(z; \lambda)$.

From the properties of the Noetherian rings there is a minimal basis of the ideal $\mathcal{I}$ of finite cardinality $m \geq 1$ formed by an initial string of averaged functions. We denote such minimal basis by

$$\{f_{j_1}(z; \lambda), \ldots, f_{j_m}(z; \lambda)\},$$

where $j_i \in \mathbb{N}$ are ordered as $\ell \leq j_1 < j_2 < \cdots < j_m$. It is clear that the ideal $\mathcal{I}$ could be minimally generated by a number of elements in $\mathbb{R}\{z, \lambda\}_{(z_0,\lambda^*)}$ less than m. But we abuse of notation and when we write a minimal basis B of $\mathcal{I}$ we mean a basis whose elements are averaged functions selected as follows:

(a) initially set $B = \{f_{j_1}\}$, where f_{j_1} is the first non-zero element of B;
(b) sequentially check successive elements f_r, starting with $r = j_1 + 1$ and ending with $r = j_m$, adjoining f_r to B if and only if $f_r \notin \langle B \rangle$, the ideal generated in $\mathbb{R}\{z, \lambda\}_{(z_0,\lambda^*)}$ by B.

If we denote by $\mathbf{V}_{\mathbb{R}}(\mathcal{I})$ the real variety of the common zeros of all the functions of the ideal $\mathcal{I}$, then clearly the infinite-type point $(z_0, \lambda^*) \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$. In particular, $d(z_0, \lambda^*, \varepsilon) \equiv 0$ for all $|\varepsilon| \ll 1$ which means that equation (1) with $\lambda = \lambda^*$ has a T-periodic solution starting at the fixed initial condition z_0 for all $|\varepsilon| \ll 1$.

We remark that typical points $(z_0, \lambda^*) \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$ are just the equilibrium points z_0 of the differential equation (1) with $\lambda = \lambda^*$, i.e., the points $z_0 \in \Omega$ such that $F(t, z_0; \lambda^*, \varepsilon) = 0$ for all $t \in \mathbb{R}$ and $|\varepsilon| \ll 1$. See the forthcoming Hopf bifurcation section for more details.

1.2. **Main results.** Now we present our main results. The proof of the following theorem is inspired in the seminal Bautin's work [1] about Hopf bifurcations from focus and centers of planar quadratic polynomial vector fields where the role of the Poincaré-Liapunov quantities is played now by the averaged functions.

Theorem 7. *Let $d(z, \lambda, \varepsilon) = \sum_{i \geq 1} f_i(z; \lambda)\varepsilon^i$ be the displacement map associated to a differential equation* (1) *and let $\ell \geq 1$ be the first subindex such that the function $f_\ell(z; \lambda^*) \not\equiv 0$ for some fixed parameter value λ^*. Assume that $z_0 \in \Omega$ is a multiple zero of the function $f_\ell(.; \lambda^*)$. Let M be an upper bound of the number of (either positive or negative) isolated branches of the T-periodic solutions that the differential equation* (1) *with $\lambda = \lambda^*$ and $|\varepsilon| \ll 1$ can have bifurcating from z_0. Then the following statements hold.*

(i) *If (z_0, λ^*) is of finite-type with order $k \geq 1$ then $M = 2k$. When k is odd then the number of such branches is also odd and $M \geq 1$.*
(ii) *If (z_0, λ^*) is of infinite-type then $M = 2(m - 1)$, where m is the cardinality of the minimal basis of the ideal $\mathcal{I} = \langle f_i(z; \lambda) : i \in \mathbb{N} \rangle$ in the ring $\mathbb{R}\{z, \lambda\}_{(z_0, \lambda^*)}$.*

Theorem 7 is proved in subsection §5.5. Of course, statement (i) of Theorem 7 is just Theorem 5 and we include it for completeness. It is interesting to compare statement (ii) of Theorem 7 with the first part of Theorem 12.25 (page 211) in the book [8]. There $\mathcal{I}$ is called the Bautin ideal of the real analytic germ $d(z, \lambda, .)$ and the Bautin depth of $\mathcal{I}$ is just $m - 1$, which coincides with the upper bound on the real cyclicity of the family of germs near (z_0, λ^*).

When $z_0 \in \Omega$ is a zero of $f_\ell(.; \lambda^*)$, from the proof of Theorem 7 it follows that for (z, λ) sufficiently close to (z_0, λ^*) and for $|\varepsilon| \ll 1$, the displacement map $d(z; \lambda, .)$ can have at most either k or $m-1$ small isolated (either positive or negative) zeros depending on the nature of the point (z_0, λ^*). This is the reason why (see subsection §1.3 and examples) we can also work with families of differential equations varying also λ and not only perturbing with the small parameter ε. In particular, for $\lambda = \lambda^*$, there are at most either k or $m - 1$ (either positive or negative) functions $\varepsilon_j^*(., \lambda^*)$ defined on a half-neighborhood of z_0 such that $d(z, \lambda^*, \varepsilon_j^*(z, \lambda^*)) \equiv 0$ for all z in that half-neighborhood. Therefore, joining the above two half-neighborhoods we obtain the $2k$ or $2(m - 1)$ upper bound of (either positive or negative) isolated branches of T-periodic solutions showed in Theorem 7.

Remark 8. First we note that the classification of singularities of smooth maps becomes more complicated as its codimension increases. For example (in one state variable) there are eleven singularities of codimension three or less which are called elementary singularities, see for example [7]. Second the branching theory [17] cannot be used to analyze infinite-type points $(z_0, \lambda^\dagger)$ when the minimum multiplicity of z_0 as root of all the averaged functions $f_j(.; \lambda^\dagger)$ for all $j \geq \ell$ is unknown. In such cases Theorem 7(ii) still can be applied to produce upper bounds on the number of the locally invertible branches. In fact, infinite-type points can be reduced (once we know a minimal basis of the ideal $\mathcal{I}$) to a finite-type, see the second part of the proof of Theorem 3 in §5.3.

Remark 9. We consider a point $(z_0, \lambda^*) \in \Omega \times \mathbb{R}^p$ of infinite-type. Since the associate ideal $\mathcal{I}$ is an ideal in the ring $\mathbb{R}\{z, \lambda\}_{(z_0, \lambda^*)}$, it is clear that $\mathcal{I}$ depends on the point (z_0, λ^*). Consequently, $\mathcal{I}$ and m also depend on (z_0, λ^*). Until now we have analyzed just one point (z_0, λ^*) of infinite-type and we have not used any notation taking care of such a dependence. In the rest of the paper we can have the situation that, for some λ^*, the function $f_\ell(., \lambda^*)$ can have several zeros $z_r \in \Omega$ and all the points (z_r, λ^*) can be of infinite-type for all the subscripts r. In this case we will use the notation $\mathcal{I}_{(z_r,\lambda^*)}$ and $m(z_r, \lambda^*)$ instead of simply $\mathcal{I}$ or m, respectively.

Remark 10. We explain with two simple examples how the singularity theory of smooth maps can be applied according with Remark 4. The following normal forms $\hat{\Delta}(z, \varepsilon)$ of the reduced displacement map $\Delta(z, \varepsilon)$ for some fixed parameters λ are characterized under strongly equivalency (that is, with $\Lambda(\varepsilon) = \varepsilon$), see for example [7]:

(i) $\hat{\Delta}(z, \varepsilon) = \delta_1 z^{\bar{k}} + \delta_2 \varepsilon$ if and only if z_0 has arbitrary multiplicity $\bar{k} \geq 2$ and is of finite-type with order $k = 1$. Here, $\delta_1 = \operatorname{sgn}(f_\ell^{(\bar{k})}(z_0)) \neq 0$ and $\delta_2 = \operatorname{sgn}(f_{\ell+1}(z_0)) \neq 0$.

(ii) $\hat{\Delta}(z, \varepsilon) = z(\delta_1 z^{\bar{k}-1} + \delta_2 \varepsilon)$ if and only if z_0 has arbitrary multiplicity $\bar{k} \geq 3$ and is a simple zero of $f_{\ell+1}(z)$. Here, $\delta_1 = \operatorname{sgn}(f_\ell^{(\bar{k})}(z_0)) \neq 0$ and $\delta_2 = \operatorname{sgn}\left(f'_{\ell+1}(z_0)\right) \neq 0$. Notice that z_0 can be of finite-type with order $k \geq 2$ or of infinite-type for the map Δ although the origin is of infinite-type for $\hat{\Delta}$.

Remark 11. We emphasize that when z_0 is an equilibrium of the differential equation (1) for any small ε then its associated trivial constant branch $z^*(\lambda, \varepsilon) = z_0$ is counted in the number of isolated branches bifurcating from z_0 when we use the multiplicity bound $\bar{k}$ of Theorem 2(ii), or when we use either the singularity theory [7], or the branching theory [17]. But the constant branches are not counted in the bounds of Theorem 7.

The following result is a straightforward consequence of Theorems 2(i) and 7 when for a fixed λ^* the zero set $f_\ell^{-1}(0) = \{z_0 \in \Omega : f_\ell(z_0, \lambda^*) = 0\}$ has finite cardinality.

Corollary 12. *Let $d(z, \lambda, \varepsilon) = \sum_{i \geq 1} f_i(z; \lambda)\varepsilon^i$ be the displacement map associated to a differential equation* (1) *and let $\ell \geq 1$ be the first subindex such that the function $f_\ell(z; \lambda^*) \not\equiv 0$ for some fixed parameter value λ^*. Assume that the set of real zeros of the function $f_\ell(.; \lambda^*)$ in Ω is finite and given by s simple zeros, m_f multiple zeros of finite-type with orders k_j for $j = 1, \ldots, m_f$, and m_c multiple zeros of infinite-type $\{z_1, \ldots, z_{m_c}\} \subset \Omega$. For each $r \in \{1, \ldots, m_c\}$ let $m(z_r, \lambda^*)$ be the cardinality of the minimal basis of the ideal $\mathcal{I}_{(z_r,\lambda^*)} = \langle f_i(z; \lambda) : i \in \mathbb{N}\rangle$ in the ring $\mathbb{R}\{z, \lambda\}_{(z_r,\lambda^*)}$. Then, for $|\varepsilon| \ll 1$, the number of (either positive or negative) isolated branches of the T-periodic solutions that differential equation* (1) *with $\lambda = \lambda^*$ can have bifurcating from a finite point is at most $s + \sum_{i=1}^{m_f} 2k_i + \sum_{r=1}^{m_c} 2(m(z_r, \lambda^*) - 1)$.*

Joining Corollary 12 and Theorem 2(ii) we obtain the following result.

Corollary 13. *Under the hypotheses of Corollary* 12, *let $\bar{k}_i$ be the multiplicity of each multiple zero of the function $f_\ell(.; \lambda^*)$ for $i = 1, \ldots, m_f + m_c$. Define for $i = 1, \ldots, m_f$ and for $j = 1, \ldots, m_c$ the integers $\bar{m}_i^f = \min\{\bar{k}_i, 2k_i\}$ and $\bar{m}_j^c = \min\{\bar{k}_j, 2(m(z_j, \lambda^*) - 1)\}$. Then, for $|\varepsilon| \ll 1$, the number of (either positive or* *negative) isolated branches of the T-periodic solutions that differential equation* (1) *with $\lambda = \lambda^*$ can have bifurcating from a finite point is bounded by*

$$s + \sum_{i=1}^{m_f} \bar{m}_i^f + \sum_{j=1}^{m_c} \bar{m}_j^c.$$

1.3. **The averaged cyclicity.** From now on we will deal with families of differential equations (1) and not with a unique member of the family as until now. So we do not fix the parameters of the family and we allow that λ varies in $\mathbb{R}^p$.

We define the *averaged cyclicity* of the full family of differential equations (1) as the maximum number of (either positive or negative) isolated branches of T-periodic solutions bifurcating from points in Ω, that is, coming from the zeros of the function $f_\ell(.;\lambda)$, defined in the statement of Lemma 1, when $|\varepsilon| \ll 1$, for any value of the parameters $\lambda \in \mathbb{R}^p$, and any initial condition $z_0 \in \Omega$. We will denote such a number as $\mathrm{Cyc}^T(F_\lambda)$, and we can compute under some finiteness assumptions an upper bound of it as follows.

First, for a fixed $j \in \mathbb{N}$ we define the open region $\Omega_j \times \Lambda_j \subset \Omega \times \mathbb{R}^p$ such that its points (z_0, λ_0) are characterized by the existence of a neighborhood $U_{(z_0,\lambda_0)} \subset \Omega_j \times \Lambda_j$ of the point (z_0, λ_0) where j is the smallest subindex such that $f_j(z;\lambda) \not\equiv 0$ for all $(z,\lambda) \in U_{(z_0,\lambda_0)}$. Observe that with this definition there can exist points $(z_c, \lambda_c) \in \Omega_j \times \Lambda_j$ which are of infinite-type because $f_i(z_c;\lambda_c) = 0$ for all $i \in \mathbb{N}$. The number of these points (z_c, λ_c) can be finite or not and also they can be isolated or not. We denote by $\Omega^* \times \Lambda^* \subset \Omega \times \mathbb{R}^p$ the set of points of infinite-type, i.e.,

$$\Omega^* \times \Lambda^* = \{(z_0, \lambda_0) \in \Omega \times \mathbb{R}^p : f_j(z_0;\lambda_0) = 0 \text{ for all } j \in \mathbb{N}\}.$$

Note that $(\Omega_j \times \Lambda_j) \cap (\Omega^* \times \Lambda^*)$ can be nonempty, but always $(\Omega_j \times \Lambda_j) \cap (\Omega_i \times \Lambda_i) = \emptyset$ if $i \neq j$.

We claim that there are finitely many possible indices j of the sets $\Omega_j \times \Lambda_j$. More precisely, $1 \leq j \leq \nu < \infty$ due to the fact that the ideals $\mathcal{I}_{(z_r,\lambda^*)}$ are finitely generated by the Hilbert basis theorem. Actually $\nu = \max_r\{m(z_r, \lambda)\}$.

On the other hand, it is clear that all the solutions of equation (1) with $\lambda \in \Lambda^*$ and initial condition $z \in \Omega^*$ are T-periodic for all $|\varepsilon| \ll 1$. In particular, if we have that the cardinalities $\#(\Lambda^*) \geq 1$ and $\#(\Omega^*) = \infty$, then there are infinitely many T-periodic solutions of equation (1) with $\lambda \in \Lambda^*$ for any $|\varepsilon| \ll 1$.

Finally we shall apply Corollary 12 to each component $\Omega_j \times \Lambda_j$ starting from $j = 1$ until $j = \nu$, assuming that $\#\{(\Omega_j \times \Lambda_j) \cap (\Omega^* \times \Lambda^*)\} = m_c^{[j]} < \infty$, and that the number of points in $\Omega_j \times \Lambda_j$ which are of finite-type is $m_f^{[j]} < \infty$. In this way we obtain, for each j, a finite bound M_j on the number of (either positive or negative) isolated branches of T-periodic solutions of equation (1) with $\lambda \in \Lambda_j$ having initial condition $z_0 \in \Omega_j$ under the hypothesis that $\bigcup_{i=1}^{j-1}(\Omega_i \times \Lambda_i) = \emptyset$.

1.4. **The algorithm for computing the averaged cyclicity.**

(i) Calculate the set $\Omega_{j_1} \times \Lambda_{j_1}$ where the function $f_{j_1}(z;\lambda)$ is not identically zero and the subindex j_1 is minimum.

(ii) Compute the zero-set of the function $f_{j_1}(.;\lambda)$ on $\Omega_{j_1}\times\Lambda_{j_1}$ given by

$$f_{j_1}^{-1}(0)=\{(z_0(\lambda),\lambda)\in\Omega_{j_1}\times\Lambda_{j_1}: f_{j_1}(z_0(\lambda),\lambda)=0\}.$$

We continue assuming the finite cardinality $\#(f_{j_1}^{-1}(0))$ of the zero-set $f_{j_1}^{-1}(0)$.

(iii) Separate, for each $\lambda\in\Lambda_{j_1}$, the simple and the multiples zeros in $f_{j_1}^{-1}(0)$. Thus we define

$$\begin{aligned}\mathcal{S}_{j_1}^{\lambda} &= \{(z_0(\lambda),\lambda)\in f_{j_1}^{-1}(0):\frac{\partial}{\partial z}f_{j_1}(z_0(\lambda),\lambda)\neq 0\},\\ \mathcal{M}_{j_1}^{\lambda} &= \{(z_0(\lambda),\lambda)\in f_{j_1}^{-1}(0):\frac{\partial}{\partial z}f_{j_1}(z_0(\lambda),\lambda)= 0\},\end{aligned}$$

and also the cardinals $\#(\mathcal{S}_{j_1}^{\lambda})=s^{[\lambda,j_1]}$ for all $\lambda\in\Lambda_{j_1}$.

(iv) For each $\lambda\in\Lambda_{j_1}$, consider the sets of points of infinite-type $\mathcal{C}_{j_1}^{\lambda}=\mathcal{M}_{j_1}^{\lambda}\cap(\Omega^*\times\Lambda^*)$ and of finite-type $\mathcal{F}_{j_1}^{\lambda}=\mathcal{M}_{j_1}^{\lambda}\backslash(\Omega^*\times\Lambda^*)$ with finite cardinalities $\#(\mathcal{C}_{j_1}^{\lambda})=m_c^{[\lambda,j_1]}$ and $\#(\mathcal{F}_{j_1}^{\lambda})=m_f^{[\lambda,j_1]}$, respectively.

(v) Compute the order $k_i^{[\lambda,j_1]}$ of all the points in $\mathcal{F}_{j_1}^{\lambda}$ for $i=1,\ldots,m_f^{[\lambda,j_1]}$.

(vi) For any point $(z_i,\lambda_i)\in\mathcal{C}_{j_1}^{\lambda}$, compute a minimal basis of the ideal $\mathcal{I}_{(z_i,\lambda_i)}$ and denote its cardinality by $m(z_i,\lambda_i)$.

(vii) Then, the averaged cyclicity $\mathrm{Cyc}^T(F_\lambda)$ of family (1) in $\Omega_{j_1}\times\Lambda_{j_1}$ is finite and bounded by $\mathrm{Cyc}^T(F_\lambda)\leq M_{j_1}$ where

$$M_{j_1}=\max_{\lambda\in\Lambda_{j_1}}\left\{s^{[\lambda,j_1]}+\sum_{i=1}^{m_f^{[\lambda,j_1]}}2k_i^{[\lambda,j_1]}+\sum_{i=1}^{m_c^{[\lambda,j_1]}}2\left(m(z_i,\lambda_i)-1\right)\right\}<\infty.$$

(viii) Repeat from step (i) until step (vii) changing j_1 by the next subindex j_i for $i=2,\ldots,\nu$, assuming that the finiteness condition in step (ii) holds in all the repetitions, that is, $\#(f_{j_i}^{-1}(0))<\infty$ for all $\lambda\in\Lambda_{j_i}$ and any i.

(ix) Finally we get an upper bound M for the averaged cyclicity $\mathrm{Cyc}^T(F_\lambda)$ of the full family of differential equations (1) in $\Omega\times\mathbb{R}^p$ given by

$$\mathrm{Cyc}^T(F_\lambda)\leq M=\max_{1\leq i\leq\nu}\{M_{j_i}\}<\infty.$$

We note that $\mathrm{Cyc}^T(F_\lambda)$ can be unbounded when $\#(f_{j_i}^{-1}(0))=\infty$ for some admissible i. This is the case when $\#(\Lambda^*)\geq 1$ and $\#(\Omega^*)=\infty$ producing infinitely many T-periodic solutions of equation (1) with $\lambda\in\Lambda^*$ for any $|\varepsilon|\ll 1$.

2. Hopf bifurcation in the plane

Consider a family of polynomial planar vector fields

$$\dot{x}=-y+P(x,y;\lambda),\quad \dot{y}=x+Q(x,y;\lambda), \tag{4}$$

with nonlinearities P and Q and parameters λ. Introducing the rescaling $(x,y)\mapsto(x/\varepsilon,y/\varepsilon)$ and next taking polar coordinates $x=r\cos\theta$, $y=r\sin\theta$, family (4) can be written near the origin as

$$\frac{dr}{d\theta}=\mathcal{F}(\theta,r;\lambda,\varepsilon), \tag{5}$$

with $\mathcal{F}(\theta, r; \lambda, 0) \equiv 0$. That is, equation (5) is written in the standard form (1) of the averaging theory with period $T = 2\pi$. Notice that the differential equation (5) is defined on the cylinder $\{(r, \theta) \in \Omega \times \mathbb{S}^1\}$ with $\Omega \subset \mathbb{R}$ an open interval containing the origin and $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$.

Remark 14. In special cases the set $\Omega^* \times \Lambda^* \subset \Omega \times \mathbb{R}^p$ of points of infinite-type have the associated ideal $\mathcal{I}$ independent on the specific point $(z_0, \lambda^*) \in \Omega^* \times \Lambda^*$ that we choose. This phenomena occurs when the sequence $\{f_j(z; \lambda)\}_{j \in \mathbb{N}} \subset \mathbb{R}[z, \lambda]$ is polynomial. Under this hypothesis $\mathcal{I}$ is a polynomial ideal in the ring $\mathbb{R}[z, \lambda]$. Therefore $\Omega^* \times \Lambda^* = \mathbf{V}_{\mathbb{R}}(\mathcal{I})$, and we have a unique value of m independently of the point of infinite-type that we consider.

If we expand the displacement map $d(z, \lambda, 1)$ of (5) with $\varepsilon = 1$ in powers of z we obtain $d(z, \lambda, 1) = \sum_{i \geq 1} v_i(\lambda) z^i$ where the coefficients $v_i \in \mathbb{R}[\lambda]$ are called the *Poincaré-Liapunov constants* associated to the equilibrium point localized at the origin of coordinates of the differential system (4). The *Bautin ideal* $\mathcal{B} \subset \mathbb{R}[\lambda]$ associated to the origin of family (4) is defined as $\mathcal{B} = \langle v_i(\lambda) : i \in \mathbb{N} \rangle$. The *center variety* is defined as $\mathbf{V}_{\mathbb{R}}(\mathcal{B}) \subset \mathbb{R}^p$, and it follows that system (4) with $\lambda = \lambda^c$ has a center at the origin if and only if $\lambda^c \in \mathbf{V}_{\mathbb{R}}(\mathcal{B})$. Now we point out a relation between the Bautin ideal $\mathcal{B}$ and the ideal $\mathcal{I}$ in the particular case of homogeneous nonlinearities of degree 2 or 3. We note that both ideals are polynomial ideals.

Proposition 15. *Let $d(z, \lambda, \varepsilon) = \sum_{j \geq 1} f_j(z; \lambda)\, \varepsilon^j$ be the displacement map associated to equation (5) coming from system (4) in the particular case that the perturbation field (P, Q) is homogeneous of degree 2 or 3. Then $f_1(z; \lambda) \equiv 0$ and $f_j(z; \lambda) = P_j(\lambda) z^{j+1}$ where $P_j \in \mathbb{R}[\lambda]$ for all $j \in \mathbb{N}$, that is, the j-th Poincaré-Liapunov constant is $v_j(\lambda) = P_{j-1}(\lambda)$. In particular, the Bautin ideal $\mathcal{B}$ and the ideal $\mathcal{I}$ have the same cardinality in their respective minimal basis. More precisely, if $\{P_{i_1}, \ldots, P_{i_m}\}$ is a minimal basis of $\mathcal{B}$, then $\{f_{i_1}, \ldots, f_{i_m}\}$ is a minimal basis of $\mathcal{I}$.*

Proof. The structure $f_j(z; \lambda) = P_j(\lambda) z^{j+1}$ where $P_j \in \mathbb{R}[\lambda]$ for all $j \in \mathbb{N}$ is easy to check for the differential equation (5). So we will prove only the second part of the proposition.

Since $\mathcal{B} = \langle P_i(\lambda) : i \in \mathbb{N} \rangle$, let $\{P_{i_1}, \ldots, P_{i_m}\}$ be a minimal basis of $\mathcal{B}$ with cardinality m. For any $j \geq i_m$ we have $f_j(z; \lambda) = P_j(\lambda) z^{j+1}$ and, since $P_j \in \mathcal{B}$, there are polynomials $q_k(\lambda)$ such that $f_j(z; \lambda) = z^{j+1} \sum_{k=1}^m q_k(\lambda) P_{i_k}(\lambda)$. Clearly this can be rewritten as $f_j(z; \lambda) = \sum_{k=1}^m r_k(z, \lambda) f_{i_k}(z, \lambda)$ just taking $r_k(z, \lambda) = q_k(\lambda) z^{j - i_k} \in \mathbb{R}[z, \lambda]$. Thus $f_j \in \langle f_{i_1}, \ldots, f_{i_m} \rangle$ for all $j \geq i_m$. □

Note that the polynomial differential system (4) has a center at the origin when $\lambda = \lambda^c \in \mathbf{V}_{\mathbb{R}}(\mathcal{B})$ if and only if $d(z, \lambda^c, \varepsilon) \equiv 0$ for all (z, ε) in a neighborhood of $(0, 0)$ or, equivalently when the functions $f_j(z; \lambda^c) \equiv 0$ for all z near the origin and all $j \in \mathbb{N}$. Clearly, for the values λ^c of the parameters we get that all the non-isolated points (z, λ^c) are of infinite-type for any $z \in \Omega$.

As usual $\mathbb{R}^+$ denotes the set of positive real numbers.

Remark 16. By construction one has $\mathcal{F}(\theta, 0; \lambda, \varepsilon) \equiv 0$, that is, $r = 0$ corresponds to the equilibrium point localized at the origin of coordinates for the full family of differential systems (4). Then it follows that $d(0,\lambda,\varepsilon) \equiv 0$ for all $\lambda \in \mathbb{R}^p$ and any $|\varepsilon| \ll 1$ or, in other words, $(z_0,\lambda) = (0,\lambda) \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$. From Proposition 15 we see that $z_0 = 0$ is always a multiple zero of $f_\ell(.;\lambda)$ for any $\lambda \in \mathbb{R}^p$. But $z_0 = 0$ has associated a finite multiplicity $\bar{k} \geq 2$ only in case that $\lambda \notin \mathbf{V}_{\mathbb{R}}(\mathcal{B})$. In this scenario, each nontrivial isolated periodic solution that system (4) has near the origin (called *small amplitude limit cycles*) corresponds exactly with an isolated (either positive or negative) branch $z^*(\lambda,.)$ of 2π-periodic solutions of equation (5) bifurcating from $z_0 = 0$ with the additional restriction that $z^*(\lambda,\varepsilon) \in \mathbb{R}^+$, because the initial conditions for system (5) must be positive. This last constraint implies that, after calculate m, we finally obtain a bound of $m-1$ for the number of locally invertible branches of small amplitude limit cycles of (4) instead of the usual bound $2(m-1)$ that gives Theorem 7(ii). The reason is that we only need to count the number of functions $\varepsilon_j^*(.,\lambda)$ defined on the positive half-neighborhood of $z_0 = 0$ such that $d(z,\lambda^*,\varepsilon_j^*(z,\lambda^*)) \equiv 0$ for all z in that positive half-neighborhood.

The *cyclicity* of the origin of the family of polynomial differential systems (4) is the maximum number of small amplitude limit cycles that can bifurcate from the singularity at the origin of that family. We denote such a cyclicity by $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0))$. We have a method for computing, under some assumptions, an upper bound of the cyclicity $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0))$ based on Corollary 13 applied to differential equation (5) and the algorithm developed for families in subsection §1.4.

Of course, for a fixed parameter λ^*, one has $\mathrm{Cyc}(\mathcal{X}_{\lambda^*},(0,0)) = \mathrm{Cyc}^{2\pi}(\mathcal{F}_{\lambda^*})$. We analyze the case in which the homogeneous nonlinearities of system (4) have degree 2 or 3 (the cases for which both the center and the cyclicity problem associated to the singularity at the origin are completely solved).

On the other hand, when Corollary 13 is used in the Hopf bifurcation context two differences arise:

(a) The zero-set $f_\ell^{-1}(0)$ is the set of zeros z_0 of $f_\ell(.;\lambda^*)$ but restricted to $\Omega \cap (\mathbb{R}^+ \cup \{0\})$.
(b) For the differential equation (5) we have that the functions $f_i \in \mathbb{R}[z,\lambda]$ are polynomials, hence the ideal $\mathcal{I} = \langle f_i(z;\lambda) : i \in \mathbb{N}\rangle$ is a polynomial ideal in the ring $\mathbb{R}[z,\lambda]$ instead of the ring $\mathbb{R}\{z,\lambda\}_{(z_0,\lambda^*)}$. See Remark 14.

2.1. **Quadratic systems.** Consider the quadratic polynomial differential systems (simply quadratic systems in what follows) in the Bautin normal form

$$\begin{array}{rcl} \dot{x} & = & -y + P(x,y;\lambda) = -y - A_3x^2 + (2A_2 + A_5)xy + A_6y^2, \\ \dot{y} & = & x + Q(x,y;\lambda) = x + A_2x^2 + (2A_3 + A_4)xy - A_2y^2, \end{array} \tag{6}$$

hence $\lambda = (A_2,A_3,A_4,A_5,A_6) \in \mathbb{R}^5$.

It is well known, see the seminal work [1] and also [18] that $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0)) = 2$ when we consider any $\lambda \in \mathbb{R}^5$. Now we want to bound the averaged cyclicity $\mathrm{Cyc}^{2\pi}(\mathcal{F}_\lambda)$ for the quadratic family (6) using our theory to compare with $\mathrm{Cyc}(\mathcal{X}_\lambda,(0,0))$.

Using Theorem 22 of the appendix we compute the functions $f_j(z;\lambda)$ associated to the differential equation (5). Next, we let $\tilde{f}_j \equiv f_j(\mod \mathcal{I}_{j-1})$ where $\mathcal{I}_j = \langle f_1(z;\lambda),\ldots,f_j(z;\lambda)\rangle$. Thus $\tilde{f}_j$ denotes the remainder of f_j upon division by a

Gröbner basis of the ideal generated by the previous f_j. Unless multiplicative constants we get

$$f_2(z;\lambda) = P_2(\lambda)z^3,\ \tilde{f}_3(z;\lambda) \equiv 0,\ \tilde{f}_4(z;\lambda) = P_4(\lambda)z^5,\ \tilde{f}_5(z;\lambda) \equiv 0,\ \tilde{f}_6(z;\lambda) = P_6(\lambda)z^7.$$

where

$$\begin{array}{rcl} P_2(\lambda) & = & A_5(A_3 - A_6), \\ P_4(\lambda) & = & A_2A_4(A_3 - A_6)(5A_3 + A_4 - 5A_6), \\ P_6(\lambda) & = & A_2A_4^2(A_3 - A_6)(5A_2^2 + A_4A_6 + 5A_6^2). \end{array}$$

We remark that the ideal $\mathcal{I}_6$ for systems (6) is not radical, i.e. $\mathcal{I}_6 \neq \sqrt{\mathcal{I}_6}$. Hence, Theorem 23 of the appendix does not work for proving that $\mathcal{I}_6$ is $\mathcal{I}$. But from Bautin's work [1] it follows that the Bautin ideal $\mathcal{B}$ is equal to $\langle P_2(\lambda), P_4(\lambda), P_6(\lambda)\rangle$. From Proposition 15, we conclude that $\mathcal{I} = \mathcal{I}_6$ and consequently $m = 3$. Thus, we have averaged cyclicity bound $\mathrm{Cyc}^{2\pi}(\mathcal{F}_\lambda) \leq m - 1 = 2$, which is sharp because $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0)) = 2$, and it cannot be improved using the multiplicity $\bar{k}$ of the zero $z_0 = 0$ in any case.

2.2. **Cubic Sibirsky systems.** Consider a cubic differential system with cubic homogeneous nonlinearities and having a center with purely imaginary eigenvalues or a focus at the origin of coordinates. Following Sibirsky [15], see also [14] and the references therein, after a linear change of coordinates the system can be written in the following form

$$\begin{array}{rcl} \dot{x} & = & -y + P(x,y;\lambda) = -y + \beta x - (\omega + \Theta - a)x^3 - (\eta - 3\mu)x^2y \\ & & -(3\omega - 3\Theta + 2a - \xi)xy^2 - (\mu - \nu)y^3, \\ \dot{y} & = & x + Q(x,y;\lambda) = x + \beta y + (\mu + \nu)x^3 + (3\omega + 3\Theta + 2a)x^2y \\ & & f(\eta - 3\mu)xy^2 + (\omega - \Theta - a)y^3, \end{array} \tag{7}$$

where $\lambda = (\omega, \Theta, a, \eta, \mu, \xi, \nu) \in \mathbb{R}^7$ are the parameters of the family.

After Żołądek's work [19] it is known that, for system (7), $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0)) = 4$ when considering any $\lambda \in \mathbb{R}^7$. Now we want to bound the averaged cyclicity $\mathrm{Cyc}^{2\pi}(\mathcal{F}_\lambda)$ for the cubic family (7) using our theory and next we compare with $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0))$.

We compute the first elements $f_j(z;\lambda)$, after we reduce them modulo the ideal $\mathcal{I}_{j-1}$, and thus we obtain $\tilde{f}_j(z;\lambda)$. We obtain $\tilde{f}_{2i+1}(z;\lambda) \equiv 0$ for $i = 0, 1, 2, 3, 4, 5$ and, unless a multiplicative constant, the first expressions of $\tilde{f}_{2i}(z;\lambda) = P_{2i}(\lambda)z^{2i+1}$ are

$$\tilde{f}_2(z;\lambda) = \xi z^3, \quad \tilde{f}_4(z;\lambda) = a\nu z^5, \quad \tilde{f}_6(z;\lambda) = a\Theta w z^7,$$
$$\tilde{f}_8(z;\lambda) = a^2\eta\Theta z^9, \quad \tilde{f}_{10}(z;\lambda) = a^2\Theta(a^2 - 4\Theta^2 - 4\mu^2)z^{11}.$$

Let m be the cardinality of a minimal basis of the ideal $\mathcal{I}$. Unfortunately $\mathcal{I}_{10} \neq \sqrt{\mathcal{I}_{10}}$, so we cannot use Theorem 23 to obtain that $\mathcal{I}$ is $\mathcal{I}_{10}$, and therefore that m is 5. We also note that we cannot use Theorem 24, because the primary decomposition $\mathcal{I}_{10} = \mathcal{R} \cap \mathcal{N}$ is such that the point we want to analyze $(z_0, \lambda) = (0, \lambda)$ is in the variety $\mathbf{V}_{\mathbb{R}}(\mathcal{N})$. But we can use Żołądek's results in [19] from where we know that

5 is the dimension of a minimal basis of the Bautin ideal $\mathcal{B}$. Adapting this result to our framework gives that

$$\mathcal{B} = \langle P_2(\lambda), P_4(\lambda), P_6(\lambda), P_8(\lambda), P_{10}(\lambda)\rangle = \langle \xi, a\nu, a\Theta w, a^2\eta\Theta, a^2\Theta(a^2-4\Theta^2-4\mu^2)\rangle.$$

Therefore, by using Proposition 15, we conclude that $\mathcal{I} = \mathcal{I}_{10}$, so that $m = 5$. Hence the averaged cyclicity bound is $\mathrm{Cyc}^{2\pi}(\mathcal{F}_\lambda) \leq m-1 = 4$, and this bound is sharp because $\mathrm{Cyc}(\mathcal{X}_\lambda, (0,0)) = 4$.

3. Bifurcations from the period annulus

3.1. **Perturbing a linear center inside the generalized Liénard systems.** We shall study the maximum number of branches of limit cycles that can bifurcate from the periodic orbits of the period annulus of a linear center perturbed inside a class of polynomial generalized Liénard differential equations of degree 7. More specifically we analyze the perturbed system

$$\dot{x} = y, \quad \dot{y} = -x - \varepsilon\left(y\hat{f}_6(x;\lambda,\varepsilon) + \hat{g}_7(x;\lambda,\varepsilon)\right), \tag{8}$$

with

$$\hat{f}_6(x;\lambda,\varepsilon) = \sum_{i=0}^{6}(A_i + B_i\varepsilon)x^i, \quad \hat{g}_7(x;\lambda,\varepsilon) = \sum_{j=1}^{7}(C_j + D_j\varepsilon)x^i.$$

Here the parameters $\lambda \in \mathbb{R}^{28}$ are the coefficients A_i, B_i, C_j, D_j for $i = 0,\dots,6$ and $j = 1,\dots,7$. In this example, first we see how from Theorem 2(ii) we obtain a uniform bound on the number of bifurcating limit cycles, that is, either equal or sharp than the obtained using our theory. We also compare it with the ones predicted by the singularity and the branching theories.

Proposition 17. *Consider the family of quintic Liénard polynomial differential systems* (8) *(that is with $A_i = B_i = C_j = D_j = 0$ for $i = 5,6$ and $j = 6,7$) under the parameter restriction λ^* given by $A_2^2 - 8A_0A_4 = 0$ and $A_2A_4 < 0$. Then, for $|\varepsilon|$ sufficiently small, limit cycle bifurcations in the period annulus of the linear center can only be produced from the periodic orbit $x^2 + y^2 = -A_2/A_4$. Moreover, the maximum number of such bifurcating branches of limit cycles is bounded by* 2 *and, if additionally $P(\lambda^*) = 768A_4^3B_0 - 192A_2A_4^2(B_2 - A_1C_2) + 84A_2^3A_3C_4 + A_2^2A_4(96B_4 - 160A_3C_2 + 3A_2C_3 - 96A_1C_4) \neq 0$, then exactly* 2 *(either positive or negative) branches bifurcate.*

Proof. Taking polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, and observing that $\dot{\theta} = -1 + \mathcal{O}(\varepsilon)$, system (8) can be written as $dr/d\theta = \mathcal{F}(\theta, r;\lambda,\varepsilon)$ with $\mathcal{F}(\theta, r;\lambda, 0) \equiv 0$. This differential equation is defined on the cylinder $\{(r,\theta) \in \Omega\times\mathbb{S}^1\}$ with $\Omega \subset \mathbb{R}$ and $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Thus we can apply to it the averaging theory with period $T = 2\pi$. Now we will estimate the averaged cyclicity $\mathrm{Cyc}^{2\pi}(\mathcal{F}_\lambda)$ of systems (8) in $\Omega \times \Lambda$ with $\Lambda = \mathbb{R}^{19}$ and $\Omega = \mathbb{R}^+$. We recall that the perturbation of a linear center by a polynomial vector field is another situation where the ideal $\mathcal{I} = \langle f_i(z;\lambda) : i \in \mathbb{N}\rangle$ is polynomial.

Computations show that $f_1(z;\lambda) = \pi z(8A_0 + 2A_2z^2 + A_4z^4)/8 \not\equiv 0$. By assumptions $A_2A_4 < 0$ and the discriminant $\Delta = A_2^2 - 8A_0A_4 = 0$. Then the function $f_1(.;\lambda^*)$ only has one positive zero $z_0 = \sqrt{-A_2/A_4} > 0$, with multiplicity $\bar{k} = 2$. Therefore, by Lemma 1, the bifurcations in the period annulus are only possible from the periodic orbit $x^2 + y^2 = z_0^2$ of the unperturbed linear system and, by Theorem 2(ii), at most can bifurcate $\bar{k} = 2$ limit cycles.

Further computations show that

$$\begin{aligned} f_2(z;\lambda) \quad = \quad & \frac{\pi}{3840} z(3840B_0 - 1920A_0C_1 + 1920A_0^2\pi - 1280A_0C_2z \\ & +960B_2z^2 - 480A_2C_1z^2 - 960A_1C_2z^2 - 2640A_0C_3z^2 \\ & +1920A_0A_2\pi z^2 + 320A_2C_2z^3 - 2304A_0C_4z^3 + 480B_4z^4 \\ & -240A_4C_1z^4 - 800A_3C_2z^4 - 600A_2C_3z^4 - 480A_1C_4z^4 \\ & +360A_2^2\pi z^4 + 1440A_0A_4\pi z^4 + 480A_4C_2z^5 - 192A_2C_4z^5 \\ & -285A_4C_3z^6 - 420A_3C_4z^6 + 480A_2A_4\pi z^6 + 96A_4C_4z^7 \\ & +150A_4^2\pi z^8). \end{aligned}$$

Therefore $f_2(z_0;\lambda) = \dfrac{z_0}{768A_4^3}P(\lambda)$. So $f_2(z_0;\lambda^*) \neq 0$ if $P(\lambda^*) \neq 0$, and $f_2(z_0;\lambda^*) = 0$ otherwise. In the first case the point (z_0,λ^*) is of finite-type with order 1, while in the second case it can be either of finite-type with order $k \geq 2$, or of infinite-type. Anyway, if (z_0,λ^*) was of infinite-type, we claim that the cardinality m of a minimal basis of the ideal $\mathcal{I} = \langle f_i(z;\lambda) : i \in \mathbb{N}\rangle$ in the ring $\mathbb{R}[z,\lambda]$ is $m \geq 3$ finishing the proof after using Corollary 13.

To prove the claim first we check that, defining $\mathcal{I}_j = \langle f_i(z;\lambda) : 1 \leq i \leq j \in \mathbb{N}\rangle$, it follows $f_2 \notin \mathcal{I}_1$ and $f_3 \notin \mathcal{I}_2$. This means that $f_j(z;\lambda)$ with $j = 1,2,3$ are elements of a minimal basis of the ideal $\mathcal{I}$ and therefore $m \geq 3$. We have make these computations with polynomial ideals in the ring $\mathbb{R}[z,\lambda]$ with $\lambda = (A_0,A_1,A_2,A_3,A_4,B_0,B_1,B_2,B_3,B_4,C_1,C_2,C_3,\ C_4,D_1,D_2,D_3,D_4)$ and introducing the discriminant Δ as an additional generator in each $\mathcal{I}_j$.

To finish, we will use the singularity theory (in particular part (i) of Remark 10 with $\bar{k} = 2$). Under the additional restriction that $P(\lambda^*) \neq 0$, so that $k = 1$ and we deduce that the branches are diffeomorphic to the branches at the origin of the normal form $\hat{\Delta}(z,\lambda^*) = \delta_1 z^2 + \delta_2\varepsilon$, and the proof is done. On the other hand, we can also apply the branching theory: since $\bar{k} = 2$ and $k = 1$ the descending section of the associated Newton's diagram only has one edge with endpoints $(0,2)$ and $(1,0)$, so that there is exactly one function $\varepsilon^*(z,\lambda^*) = \alpha(\lambda^*)(z-z_0)^2 + o((z-z_0)^2)$ with $\alpha \neq 0$ vanishing identically the displacement map. $\square$

In next example we show how the bounds obtained using the classical theory for multiple points stated in Theorem 2(ii) is improved using our results and, moreover the bound is compared with the exact number of bifurcating branches predicted by the singularity and branching theories.

Proposition 18. *Consider the family of septic Liénard polynomial differential systems* (8) *under the parameter restriction* λ^* *given by* $4A_4^2 - 15A_2A_6 = 0$, $8A_4^3 - 675A_0A_6^2 = 0$ *and* $A_4A_6 < 0$. *Then, for* $|\varepsilon|$ *sufficiently small, limit cycle bifurcations in the period annulus of the linear center can only be produced from the periodic orbit* $x^2 + y^2 = -8A_4/(15A_6)$. *Moreover, exactly 1 branch of limit cycles bifurcates when* $P(\lambda^*) = 768A_4^3B_0 - 192A_2A_4^2(B_2 - A_1C_2) + 84A_2^3A_3C_4 + A_2^2A_4(96B_4 - 160A_3C_2 + 3A_2C_3 - 96A_1C_4) \neq 0$.

Proof. As in the proof of Proposition 17, after taking polar coordinates, we write family (8) into the 2π-periodic standard form $dr/d\theta = \mathcal{F}(\theta, r; \lambda, \varepsilon)$ in $\Omega \times \Lambda$ with $\Lambda = \mathbb{R}^{26}$, $\Omega = \mathbb{R}^+$, and associated polynomial ideal $\mathcal{I} = \langle f_i(z; \lambda) : i \in \mathbb{N} \rangle$.

The first averaged function is $f_1(z; \lambda) = \pi z(64A_0 + 16A_2z^2 + 8A_4z^4 + 5A_6z^6)/64 \not\equiv 0$. The zeros of $f_1(.; \lambda)$ comes from a cubic equation for the unknown z^2. Using the discriminants of the cubic equations it follows that $f_1(.; \lambda)$ has a multiple zero $z_0 > 0$ of multiplicity 3 if and only if $\Delta_i = 0$ for $i = 1, 2, 3$, where $\Delta_1 = -4A_2^2A_4^2 + 32A_0A_4^3 + 20A_2^3A_6 - 180A_0A_2A_4A_6 + 675A_0^2A_6^2$, $\Delta_2 = 4A_4^2 - 15A_2A_6$ and $\Delta_3 = 16A_4^3 - 90A_2A_4A_6 + 675A_0A_6^2$. In particular, $z_0^2 = -\dfrac{8A_4}{15A_6}$ with our assumption $A_4A_6 < 0$. In order to simplify the polynomial conditions in the parameter space producing the unique zero $z_0 > 0$ of $f_1(.; \lambda)$ with multiplicity $\bar{k} = 3$ we calculate the resultants of each pair of polynomials Δ_i with respect to A_2 and, since $A_6 \neq 0$ we obtain the necessary condition $8A_4^3 - 675A_0A_6^2 = 0$ stated in the proposition. Solving for A_0 this condition and substituting into $\Delta_2 = \Delta_3 = 0$ produces the extra necessary condition $4A_4^2 - 15A_2A_6 = 0$ also stated in the proposition. Notice that we can solve for A_2 from the above equation.

In summary, when $|\varepsilon| \ll 1$, the bifurcation of limit cycles from the period annulus of the linear center is only possible from the circle $x^2 + y^2 = z_0^2$. By Theorem 2(ii), the maximum number of limit cycles that can bifurcate is bounded by $\bar{k} = 3$.

Now, we will use our method in order to improve the above classical bound. The next averaged function is

$$
\begin{aligned}
f_2(z; \lambda) \quad = \quad & \frac{\pi}{860160} z(860160B_0 - 430080A_0C_1 + 430080A_0^2\pi - 286720A_0C_2z + \\
& 215040B_2z^2 - 107520A_2C_1z^2 - 215040A1C_2z^2 - 591360A_0C_3z^2 + \\
& 430080A_0A_2\pi z^2 + 71680A_2C_2z^3 - 516096A_0C_4z^3 + 107520B_4z^4 - \\
& 53760A_4C_1z^4 - 179200A_3C_2z^4 - 134400A_2C_3z^4 - 107520A_1C_4z^4 - \\
& 663040A_0C_5z^4 + 80640A_2^2\pi z^4 + 322560A_0A_4\pi z^4 + 107520A_4C_2z^5 - \\
& 43008A_2C_4z^5 - 614400A_0C_6z^5 + 67200B_6z^6 - 33600A_6C_1z^6 - \\
& 156800A_5C_2z^6 - 63840A_4C_3z^6 - 94080A_3C_4z^6 - 150080A_2C_5z^6 - \\
& 67200A_1C_6z^6 + 107520A_2A_4\pi z^6 + 268800A_0A6\pi z^6 + 112000A6C_2z^7 + \\
& 21504A_4C_4z^7 - 92160A_2C_6z^7 - 38640A_6C_3z^8 - 84672A_5C_4z^8 - \\
& 70560A_4C_5z^8 - 60480A_3C_6z^8 + 33600A_4^2\pi z^8 + 84000A_2A_6\pi z^8 + \\
& 40320A_6C_4z^9 - 15360A_4C_6z^9 - 42280A_6C_5z^{10} - 55440A_5C_6z^{10} + \\
& 50400A_4A_6\pi z^{10} + 9600A_6C_6z^{11} + 18375A_6^2\pi z^{12}).
\end{aligned}
$$

Therefore one can check that $f_2(z_0; \lambda^*) \neq 0$ if and only if $P(\lambda^*) \neq 0$, where the polynomial P is displayed in the statement of the proposition. Thus, we conclude that the point (z_0, λ^*) is of finite-type with order $k = 1$ only when $P(\lambda^*) \neq 0$. In this situation, from statement (i) of Theorem 7, we know that at most 2 branches of limit cycles bifurcate.

Finally, using the singularity theory (in particular part (i) of Remark 10 with $\bar{k} = 3$) we deduce that the branches are diffeomorphic to the branches at the origin of the normal form $\hat{\Delta}(z, \lambda^*) = \delta_1 z^3 + \delta_2 \varepsilon$, and the proof is done. Of course the same conclusion holds from the branching theory: since $\bar{k} = 3$ and $k = 1$ the Newton's diagram only has one edge with endpoints $(0, 3)$ and $(1, 0)$, and therefore there is exactly one function $\varepsilon^*(z, \lambda^*) = \alpha(\lambda^*)(z - z_0)^3 + o((z - z_0)^3)$ with $\alpha \neq 0$ such that $d(z, \lambda, \varepsilon^*(z, \lambda^*)) \equiv 0$ for any z near z_0.

We emphasize that, although we will not do it, the degeneracy of the problem can be augmented by imposing $P(\lambda^*) = 0$ and solving for B_0, so that we get new (and huge) parameter restrictions in the expression of $f_3(z_0; \lambda^*)$, from where we can decide if the point (z_0, λ^*) is of finite-type with order 2. $\square$

4. The polynomial ideal $\mathcal{I}$ when the parameters are fixed

Throughout this section we will pick up just one element of family (1) by fixing its parameter, say taking $\lambda = \lambda^*$. We will also work under the hypothesis that the ideal $\mathcal{I}$ is polynomial. Indeed, following the proof of Lemma 9 in [6], we know that $\mathcal{I} = \langle f_j : j \in \mathbb{N} \rangle$ is a polynomial ideal in the ring $\mathbb{R}[z]$ when equation (1) with fixed $\lambda = \lambda^*$ is a polynomial equation, i.e., when $F_i(t, x; \lambda^*)$ are polynomial in x for all $i \in \mathbb{N}$.

Under these hypotheses we will see now that Theorem 23 is strongly simplified. We recall before that, since $\mathcal{I}$ is an ideal in the ring of univariate polynomials, $\mathcal{I}$ is a principal ideal, see for instance [3]. Thus $\mathcal{I}$ is generated by one element $\mathcal{I} = \langle g \rangle$ where $g \in \mathbb{R}[z]$ is unique up to a multiplication by a nonzero constant in $\mathbb{R}$. In fact, if $\mathcal{I} \neq \{0\}$ then the generator g is a nonzero polynomial of minimum degree contained in $\mathcal{I}$. Moreover, we note that for any $p, q \in \mathbb{R}[z]$ one has $\langle p, q \rangle = \langle r \rangle$, where $r = \gcd(p, q)$ is a greatest common divisor of p and q, see again [3]. Defining $\mathcal{I}_s = \langle f_j : 1 \leq j \leq s \in \mathbb{N} \rangle$ the ideal in $\mathbb{R}[z]$ generated by the first s averaged functions, we have the following result.

Theorem 19. *Let the ideal $\mathcal{I}_s = \langle \hat{g} \rangle \subset \mathbb{R}[z]$ where all the roots of $\hat{g}$ are real and simple. Assume the equality $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) = \mathbf{V}_{\mathbb{R}}(\mathcal{I}_s)$ of real varieties holds. Then $\mathcal{I} = \mathcal{I}_s$.*

Proof. The proof is divided in two steps.

(i) First, we claim that $\mathcal{I}_s$ is a radical ideal. To this end we recall that in the ring $\mathbb{R}[z]$ of univariate polynomials the nontrivial radical ideals are precisely those ideals generated by square-free polynomials, see [3]. In consequence, when $\hat{g} \notin \{0, 1\}$, $\mathcal{I}_s$ is radical if and only if $\hat{g}$ has no repeated roots over $\mathbb{C}$ or, equivalently $\hat{g}$ and its derivative $\hat{g}'$ are coprime. Since by hypothesis the roots of $\hat{g}$ are simple, then $\hat{g}$ is square-free and we prove claim (i).

(ii) Second, we claim that the equality $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) = \mathbf{V}_{\mathbb{C}}(\mathcal{I}_s)$ of complex varieties holds. Since the polynomial $\hat{g}$ has no non-real roots then clearly $\mathbf{V}_{\mathbb{C}}(\mathcal{I}_s) = \mathbf{V}_{\mathbb{R}}(\mathcal{I}_s)$. From our hypotheses we obtain that $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) = \mathbf{V}_{\mathbb{C}}(\mathcal{I}_s)$. Since by definition $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) \subseteq \mathbf{V}_{\mathbb{C}}(\mathcal{I})$, the former implies that

$$\mathbf{V}_{\mathbb{C}}(\mathcal{I}_s) \subseteq \mathbf{V}_{\mathbb{C}}(\mathcal{I}). \tag{9}$$

Finally, taking into account that $\mathcal{I}_s \subseteq \mathcal{I}$, one has $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \subseteq \mathbf{V}_{\mathbb{C}}(\mathcal{I}_s)$ which combined with (9) gives the proof of claim (ii).

From the former claims (i) and (ii) and Theorem 23 applied with a number of parameters $p = 0$, we conclude that $\mathcal{I} = \mathcal{I}_s$ finishing the proof. $\square$

4.1. **An Abel equation.** We consider the 2π-periodic Abel differential equation in the standard form

$$\dot{x} = \varepsilon x \sum_{i=0}^{2} A_i(t,\varepsilon)\, x^i, \tag{10}$$

defined on $\Omega = \mathbb{R}$. Let $z \in \mathbb{R}$ be the initial condition for the solutions of equation (10). Since equation (10) is polynomial in x, we know that the averaged functions $f_j(z)$ are polynomial. Hence, the ideal $\mathcal{I} = \langle f_j : j \in \mathbb{N} \rangle$ is an ideal in the ring $\mathbb{R}[z]$.

Taking certain coefficients $A_i(t,\varepsilon)$ in equation (10) in the next result we will compute the associated ideal $\mathcal{I}$, which will provide using our theory a bound to the number of nonconstant isolated branches of bifurcating 2π-periodic solutions of equation (10). We also compare this bound with the classical one, and also with the ones provided by the singularity and the branching theories.

Proposition 20. *Consider the Abel differential equation* (10) *with coefficients*

$$A_0(t,\varepsilon) = \varepsilon - 2\varepsilon^2 \cos t + \sin t, \quad A_1(t,\varepsilon) = 3\varepsilon + \cos t + \varepsilon \sin t, \quad A_2(t,\varepsilon) = 1 + \cos t + \sin t.$$

The only finite point from which, for $|\varepsilon| \ll 1$, they can bifurcate nonconstant isolated branches of 2π-periodic solutions of equation (10) *is the equilibrium at the origin. The number of such nontrivial (either positive or negative) branches is exactly either* 0 *or* 2*.*

Proof. Computing the first averaged function we obtain $f_1(z) = 2\pi z^3 \not\equiv 0$. Therefore, for $|\varepsilon| \ll 1$, the 2π-periodic solutions of the Abel equation (10) bifurcating from a finite point only can bifurcate from the equilibrium at $z_0 = 0$. Due to the fact that the multiplicity $\bar{k} = 3$ of $z_0 = 0$ is odd, we know that the number of isolated branches of 2π-periodic solutions bifurcating from $z_0 = 0$ must be either one or three, see Theorem 2(ii). But recalling Remark 11 we must take into account the trivial branch always present coming from the equilibrium at the origin of (10) for any ε. So we deduce that either zero or two (nontrivial) isolated branches of 2π-periodic solutions bifurcate from $z_0 = 0$.

Now we compute the next averaged functions yielding $f_2(z) = \pi z(2 + 5z + 2z^2 + z^3 + 6\pi z^4)$ and $f_j \in \mathcal{I}_2$ for $j = 3, \ldots, 7$, so that it is probable that $\mathcal{I}$ is just $\mathcal{I}_2$. To see that this is indeed the situation, let $\hat{g} = \gcd(f_1, f_2) = z$ be a greatest common divisor of f_1 and f_2. Then $\mathcal{I}_2 = \langle \hat{g} \rangle$ where $\hat{g}$ only has a real simple root at $z_0 = 0$. On the other hand, since $f_j(0) = 0$ for any $j \in \mathbb{N}$ because $z = 0$ is an equilibrium, it is clear that $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) = \mathbf{V}_{\mathbb{R}}(\mathcal{I}_2) = \{z = 0\}$. In conclusion, from Theorem 19, $\mathcal{I} = \mathcal{I}_2$ and therefore $\mathcal{I}$ has a minimal basis formed by averaged functions of cardinality $m = 2$. Thus, using statement (ii) of Theorem 7, at most two (either positive or negative) isolated branches of 2π-periodic solutions bifurcate from the origin. Notice that the equilibrium at the origin of equation (10) for any ε does not count in the former bound, see Remark 11.

We end noticing that $\bar{k} = 3$ and z_0 is a simple zero of $f_{\ell+1}(z) = f_2(z)$ so that, from the singularity theory (in particular part (ii) of Remark 10), the branches are diffeomorphic to the branches at the origin of the normal form $\hat{\Delta}(z,\varepsilon) = z(\delta_1 z^2 + \delta_2 \varepsilon)$ and the proof is done. Observe that although $z_0 = 0$ is of infinite-type we can still use branching theory because z_0 is a simple root of f_2. More specifically, we can factor out $\Delta(z,\varepsilon) = z\Delta^*(z,\varepsilon)$ analytically, and by applying the branching theory to $\Delta^*(z,\varepsilon)$ whose Newton's diagram only has one descending edge with endpoints $(0,2)$ and $(1,0)$, there is exactly one function $\varepsilon^*(z) = \alpha z^2 + o(z^2)$ with $\alpha \neq 0$ such that $\Delta^*(z,\varepsilon^*(z)) \equiv 0$ for any z near the origin. □

5. Proofs

First we note that the zeros of the displacement function d with $\varepsilon \neq 0$ coincide with the zeros of the reduced displacement map

$$\Delta(z,\lambda,\varepsilon) = \frac{d(z,\lambda,\varepsilon)}{\varepsilon^\ell} = f_\ell(z;\lambda) + \sum_{i\geq 1} f_{\ell+i}(z;\lambda)\varepsilon^i. \tag{11}$$

5.1. **Proof of Lemma 1.**

Proof. Since $d(z^*(\varepsilon,\lambda),\lambda,\varepsilon) \equiv 0$ for all $\varepsilon \neq 0$ in a sufficiently small half-neighborhood of zero we see that also $\Delta(z^*(\varepsilon),\lambda,\varepsilon) \equiv 0$. Thus we have $f_\ell(z^*(\varepsilon);\lambda) + O(\varepsilon) \equiv 0$, see (11). Evaluating this condition at $\varepsilon = 0$ yields $f_\ell(z_0;\lambda) = 0$. □

5.2. **Proof of Theorem 2(i).**

Proof. We will analyze the reduced displacement map (11). Since z_0 is a simple zero of $f_\ell(.;\lambda^*)$, by the Implicit Function Theorem applied to $\Delta(z,\lambda^*,\varepsilon)$ in a neighborhood of $(z,\varepsilon) = (z_0,0)$ we find a unique analytic function $z^*(\varepsilon)$ defined for $|\varepsilon| \ll 1$ and satisfying $z^*(0) = z_0$ such that $\Delta(z^*(\varepsilon),\lambda^*,\varepsilon) \equiv 0$. So Theorem 2(i) follows. □

5.3. **Proof of Theorem 3.**

Proof. In the first part we assume that $(z_0,\lambda^*) \in \Omega \times \mathbb{R}^p$ is of finite-type. Then branching theory can be used to analyze the nature and structure of the local zero-set of the reduced displacement map $\Delta(z,\lambda^*,\varepsilon) = f_\ell(z;\lambda^*) + \sum_{i\geq 1} f_{\ell+i}(z;\lambda^*)\varepsilon^i$, which is analytic near $(z,\varepsilon) = (z_0,0)$. Indeed in this case Δ fall in the called *quasi-regular* class, see for more details [17]. Then any continuous functions $\varepsilon^*(z,\lambda^*)$ satisfying $\varepsilon^*(z_0,\lambda^*) = 0$ and $\Delta(z,\lambda^*,\varepsilon^*(z,\lambda^*)) \equiv 0$ can be locally expressed as convergent either power series (therefore analytic branches) or Puiseux series determined by the descending section of the associated Newton's diagram to Δ. In any case one has that $\varepsilon^*(z,\lambda^*) = \alpha(\lambda^*)(z-z_0)^{r/s} + o(r/s)$ with $\alpha \in \mathbb{R}\backslash\{0\}$ and $0 < r/s \in \mathbb{Q}$. Thus the local behavior of $\varepsilon^*(.,\lambda^*)$ near z_0 is dominated by the leading term of this expansion. In particular the function $\varepsilon^*(.,\lambda^*)$ is locally invertible, being the branch $z^*(\lambda^*,.)$ its inverse function.

In the second part we will assume that the point (z_0,λ^*) is of infinite-type. We will see how these points can be reduced to finite-type provided a minimal basis $B_{\min} = \{f_{j_1}(z;\lambda),\ldots,f_{j_m}(z;\lambda)\}$ of the ideal $\mathcal{I}$ is known. The main step is a rearranging of the displacement function as a finite sum by adapting Lemma 12.21 of [8] or Lemma 6.1.6 of [12] to our context. More precisely, for (z,λ) sufficiently close to (z_0, λ^*) and for ε near zero, the displacement function d can be written as

$$d(z, \lambda, \varepsilon) = \sum_{i=1}^{m} f_{j_i}(z; \lambda)\, \varepsilon^{j_i} \psi_i(z, \lambda, \varepsilon), \tag{12}$$

where the functions ψ_i's are analytic and $\psi_i(z, \lambda, 0) = 1$.

Let $\beta \in \mathbb{N}$ be the minimum multiplicity of z_0 as root of all the generators $f_{j_i}(.; \lambda)$ of $B_{\min}$ for $i = 1, \ldots, m$. Then it follows that (12) has the analytic factorization $d(z, \lambda, \varepsilon) = (z - z_0)^\beta \breve{d}(z, \lambda, \varepsilon)$ where $\breve{d}(z, \lambda, \varepsilon) = \sum_{i=1}^{m} \breve{f}_{j_i}(z; \lambda)\, \varepsilon^{j_i} \psi_i(z, \lambda, \varepsilon)$ and there is at least one index $i = i^*$ such that $\breve{f}_{j_{i^*}}(z_0; \lambda) \neq 0$. Therefore now the point (z_0, λ^*) is of finite-type for the map $\breve{d}(z, \lambda, \varepsilon)$ and we can repeat the first part of the proof just changing d by $\breve{d}$. □

5.4. **Proof of Theorem 5.** Theorem 5 is just a consequence of the following result.

Proposition 21. *Assume that $f_\ell \not\equiv 0$ and there exists $z_0 \in \Omega$ such that $f_\ell(z_0; \lambda) = 0$ and $\frac{\partial}{\partial z} f_\ell(z_0; \lambda) = 0$. Assume also that there is $k \geq 1$, the minimum integer satisfying $f_{\ell+k}(z_0; \lambda) \neq 0$. Then there are at most k functions $\varepsilon_i^*(z)$ with $i = 1, \ldots, k$ where $\varepsilon_i^*(z_0) = 0$ and satisfying $\Delta(z, \lambda, \varepsilon_i^*(z)) \equiv 0$ for all z in a sufficiently small neighborhood of z_0. Moreover, if k is odd then $\varepsilon_1^*(z)$ exists.*

Proof. Taking k derivatives of Δ with respect to ε and evaluating at $(z_0, \lambda, 0)$ we obtain

$$\frac{\partial^k \Delta}{\partial \varepsilon^k}(z_0, \lambda, 0) = k!\, f_{\ell+k}(z_0; \lambda) \neq 0,$$

where we have used in the last step the hypothesis $f_{\ell+k}(z_0; \lambda) \neq 0$. Then, from the Weierstrass preparation theorem (see for instance [16]), we can factorize Δ analytically around the point $(z, \lambda, \varepsilon) = (z_0, \lambda, 0)$ as

$$\Delta(z, \lambda, \varepsilon) = P_k(z, \lambda, \varepsilon) U(z, \lambda, \varepsilon), \tag{13}$$

where $U(z_0, \lambda, 0) = 1$ and P_k is a polynomial of degree k in the variable ε given by

$$P_k(z, \lambda, \varepsilon) = f_\ell(z; \lambda) + \sum_{i=1}^{k-1} a_i(z; \lambda) \varepsilon^i + \varepsilon^k,$$

where the coefficients $a_i(z; \lambda)$ are analytic functions near $z = z_0$. Due only to the degree, it is clear that there are at most k functions $\varepsilon_i^*(z, \lambda)$ with $i = 1, \ldots, k$ where $\varepsilon_i^*(z_0, \lambda) = 0$ and satisfying $P_k(z, \lambda, \varepsilon_i^*(z)) \equiv 0$, hence $\Delta(z, \lambda, \varepsilon_i^*(z)) \equiv 0$. This proves the first part of the proposition. The last part is a straightforward consequence of the continuous dependence of the roots of polynomials on its coefficients if the degree of the polynomial does not change. □

5.5. **Proof of Theorem 7.**

Proof. Let $(z_0, \lambda^\dagger)$ be a point of finite-type with order $k \geq 1$. By definition, $f_{\ell+k}(z_0; \lambda^\dagger) \neq 0$. Then for (z, λ) close to $(z_0, \lambda^\dagger)$ we have $f_{\ell+k}(z; \lambda) \neq 0$, and consequently we can write

$$d(z, \lambda, \varepsilon) = \sum_{i=\ell}^{\ell+k-1} f_i(z; \lambda)\, \varepsilon^i + f_{\ell+k}(z; \lambda) \psi(z, \lambda, \varepsilon)\, \varepsilon^{\ell+k},$$

where the function ψ is analytic and satisfies $\psi(z, \lambda, 0) = 1$. Now we can make repeated application of the Rolle's Theorem as in the proof of Proposition 6.1.2 of [12] to see that the function d behaves like a polynomial in ε near $(z_0; \lambda^\dagger)$, hence the number of zeros of $d(z, \lambda, .)$ is bounded. More specifically, it follows that the maximum number of isolated zeros of $d(z, \lambda, .)$ coming from the zero $(z_0; \lambda^\dagger)$ of order k in the interval $(0, \hat{\varepsilon})$ with $\hat{\varepsilon} > 0$ sufficiently small is k. We note that using the same arguments we obtain that the number of isolated zeros of $d(z, \lambda, .)$ in $(-\hat{\varepsilon}, 0)$ with $\hat{\varepsilon} > 0$ sufficiently small is also bounded by k. This proves statement (i).

Now we will prove statement (ii). Let (z_0, λ^*) be a point of infinite-type and $\{f_{j_1}(z; \lambda), \ldots, f_{j_m}(z; \lambda)\}$ a minimal basis of the ideal $\mathcal{I}$ of finite cardinality $m \geq 1$ where $\ell \leq j_1 < j_2 < \cdots < j_m$. Then, as in the second part of the proof of Theorem 3, we can apply Lemma 12.21 of [8] or Lemma 6.1.6 of [12] in such a way that, for (z, λ) sufficiently close to (z_0, λ^*) and for ε near zero, the displacement function d adopts the form (12), that is,

$$d(z, \lambda, \varepsilon) = \sum_{i=1}^{m} f_{j_i}(z; \lambda)\, \varepsilon^{j_i} \psi_i(z, \lambda, \varepsilon),$$

where the functions ψ_i's are analytic and $\psi_i(z, \lambda, 0) = 1$. Again following [12] (in particular Theorem 6.1.7 of [12]) we have that $d(z; \lambda, .)$ can have at most $m - 1$ small isolated (either positive or negative) zeros, that is, there are at most $m - 1$ functions either $\varepsilon_j^*(z; \lambda) \geq 0$ or $\varepsilon_j^*(z; \lambda) \leq 0$ such that $d(z, \lambda, \varepsilon_j^*(z; \lambda)) \equiv 0$ for $j = 1, \ldots, m - 1$, for all (z, λ) sufficiently close to (z_0, λ^*). This proves statement (ii). This proof is closely related to Theorem 12.25 (page 211) of [8]. □

6. Appendix

6.1. **The expansion of the displacement map.** In order to compute the expansion in power series of ε in the displacement map $d(z, \lambda, \varepsilon) = \sum_{i \geq 1} f_i(z; \lambda)\, \varepsilon^i$, first we impose that $x(t; z, \lambda, \varepsilon)$ be a solution of equation (1), that is,

$$\frac{\partial x}{\partial t}(t; z, \lambda, \varepsilon) = \sum_{i \geq 1} F_i(t, x(t; z, \lambda, \varepsilon); \lambda)\, \varepsilon^i, \tag{14}$$

for all small enough $|\varepsilon|$. Taking into account (2), the left-hand side of (14) is expanded as

$$\frac{\partial x}{\partial t}(t; z, \lambda, \varepsilon) = \sum_{i \geq 1} \frac{\partial x_i}{\partial t}(t, z, \lambda)\, \varepsilon^i.$$

The power series in ε of the right-hand side of (14) is more involved. To get it, first we can perform the Taylor expansion of $F_i(t, x; \lambda)$ at $x = z$. Using (2) again we get

$$\begin{aligned} F_i(t, x(t; z, \lambda, \varepsilon); \lambda) &= F_i(t, z + \sum_{j \geq 1} x_j(t, z, \lambda)\, \varepsilon^j; \lambda) \\ &= F_i(t, z; \lambda) + \sum_{\alpha \geq 1} \frac{1}{\alpha!} \frac{\partial^\alpha F_i}{\partial x^\alpha}(t, z; \lambda) \left(\sum_{j \geq 1} x_j(t, z, \lambda)\, \varepsilon^j \right)^\alpha. \end{aligned}$$

Now we impose that equation (14) is verified equating all the coefficients with the same power of ε. Doing this procedure one obtains a sequence of linear differential equations for the unknown functions $x_i(t, z, \lambda)$ which can be solved with the initial conditions $x_j(0,z,\lambda)=0$. Finally we recall that $f_i(z;\lambda)=x_i(T;z,\lambda)$. In particular, the function $x_1(t,z,\lambda)$ satisfies the Cauchy problem $\dfrac{\partial x_1}{\partial t}(t,z,\lambda)=F_1(t,z,\lambda)$ with initial value $x_1(0,z,\lambda)=0$, hence $x_1(t,z,\lambda)=\int_0^t F_1(\tau,z,\lambda)\,d\tau$ and therefore

$$f_1(z;\lambda)=\int_0^T F_1(t,z,\lambda)\,dt.$$

The above algorithm can be summarized in the next result of [6].

Theorem 22. *The solution $x(t;z,\lambda,\varepsilon)$ of the T-periodic analytic equation* (1) *having initial condition $x(0;z,\lambda,\varepsilon)=z$ can be written as $x(t;z,\lambda,\varepsilon)=z+\sum_{j\geq 1}x_j(t,z,\lambda)\,\varepsilon^j$ where the $x_j(t,z,\lambda)$ can be computed recursively as follows:*

$$\begin{aligned} x_1(t,z,\lambda) &= \int_0^t F_1(\tau,z;\lambda)\,d\tau,\\ x_k(t,z,\lambda) &= \int_0^t \Bigg(F_k(\tau,z;\lambda)+\sum_{\ell=1}^{k-1}\sum_{i=1}^{\ell}\frac{1}{i!}\frac{\partial^i F_{k-\ell}}{\partial x^i}(\tau,z;\lambda)\\ &\qquad \times \sum_{j_1+j_2+\cdots+j_i=\ell}\ \prod_{p=1}^{i} x_{j_p}(\tau,z,\lambda)\Bigg)\,d\tau, \end{aligned}$$

for all $k\geq 2$, where j_m are positive integers for all $m=1,\ldots,i$.

We have used Theorem 22 to compute the averaged functions $f_i(z;\lambda)$ in all the examples of this work.

6.2. **Cyclicity bound theorems in averaging theory.** The results of this section are restricted to the case in which the ideal $\mathcal{I}$ is a polynomial ideal in the ring $\mathbb{R}[z,\lambda]$. The reason is because in the proofs of the forthcoming theorems we need to use Hilbert Nullstellensatz that relates complex varieties and ideals in $\mathbb{C}[z,\lambda]$.

The following result is useful to obtain a set of generators of the polynomial ideal $\mathcal{I}$ in case that $\mathcal{I}=\sqrt{\mathcal{I}}$, that is when $\mathcal{I}$ is a radical ideal. It is useful for analyzing the multiple zeros z_0 of $f_\ell(.;\lambda)$ with (z_0,λ) of infinite-type in Corollaries 12 and 13, in particular for the computation of the cardinality m of the minimal base of $\mathcal{I}$. We use the notation $\mathcal{I}_k=\langle f_i(z;\lambda):1\leq i\leq k\rangle$. The following theorem is proved in [5], we only state it using our notation and prove it because it has a short proof.

Theorem 23 (Radical Ideal Cyclicity Bound Theorem)**.** *Let $\mathcal{I}$ be a polynomial ideal. Assume that the equality of complex varieties $\mathbf{V}_{\mathbb{C}}(\mathcal{I})=\mathbf{V}_{\mathbb{C}}(\mathcal{I}_k)$ holds in $\mathbb{C}^{p+1}$ for some $k\in\mathbb{N}$, and that $\mathcal{I}_k$ is a radical ideal. Then $\mathcal{I}=\mathcal{I}_k$.*

Proof. Suppose that $\mathbf{V}_{\mathbb{C}}(\mathcal{I})=\mathbf{V}_{\mathbb{C}}(\mathcal{I}_k)$. From the Strong Hilbert Nullstellensatz we know that the above equality of complex varieties is equivalent to the equality of polynomial ideals $\sqrt{\mathcal{I}}=\sqrt{\mathcal{I}_k}$. You can consult for example Proposition 3.1.16 of [12]. Then, using the assumption $\sqrt{\mathcal{I}_k}=\mathcal{I}_k$, yields

$$\mathcal{I}_k\subset\mathcal{I}\subset\sqrt{\mathcal{I}}=\sqrt{\mathcal{I}_k}=\mathcal{I}_k,$$

and therefore $\mathcal{I}=\mathcal{I}_k$ finishing the proof. $\square$

Assume that $\mathcal{I} \neq \sqrt{\mathcal{I}}$. Now Theorem 23 does not work but we still can bound the number of (either positive or negative) isolated branches of the T-periodic solutions of equation (1) with $\lambda = \lambda^*$ that, for $|\varepsilon|$ sufficiently small, bifurcate from the zeros $z_0 \in \Omega$ of the averaged function $f_\ell(.;\lambda^*)$ when (z_0, λ^*) is a point of infinite-type and (z_0, λ^*) belongs to certain pieces of the variety $\mathbf{V}_\mathbb{R}(\mathcal{I})$ that we specified below. The following theorem is an adaptation to our framework of Theorem 20 in [5].

Theorem 24 (Non-Radical Ideal Cyclicity Bound Theorem)**.** *Let the ideal $\mathcal{I}$ be a polynomial ideal. Assume that the equality of complex varieties $\mathbf{V}_\mathbb{C}(\mathcal{I}) = \mathbf{V}_\mathbb{C}(\mathcal{I}_k)$ holds in $\mathbb{C}^{p+1}$ for some $k \in \mathbb{N}$ but $\mathcal{I}_k$ is not a radical ideal. Let κ be the cardinality of a minimal basis of $\mathcal{I}_k$, hence with $\kappa \leq k$. Suppose a primary decomposition of $\mathcal{I}_k$ can be written as $\mathcal{I}_k = \mathcal{R} \cap \mathcal{N}$ where $\mathcal{R}$ is the intersection of the ideals in the decomposition that are prime, and $\mathcal{N}$ is the intersection of the remaining ideals in the decomposition. Then the number of (either positive or negative) isolated branches of the T-periodic solutions that can have any equation of family* (1) *corresponding to parameters $\|\lambda - \lambda^*\| \ll 1$ bifurcating, for $|\varepsilon| \ll 1$, from a point $z_0 \in \Omega$ when $(z_0, \lambda^*) \in \mathbf{V}_\mathbb{R}(\mathcal{I}) \setminus \mathbf{V}_\mathbb{R}(\mathcal{N})$ is at most $2(\kappa - 1)$.*

A key step in the proof of Theorem 24 is to use the following result from [4] based on the arguments of Proposition 1 in [9]. For a subset $S \subset \mathbb{C}^{p+1}$, we denote by $\mathbf{I}(S)$ the ideal in the ring $\mathbb{C}[z, \lambda]$ defined by $\mathbf{I}(S) = \{g \in \mathbb{C}[z,\lambda] : g(z_0, \lambda_0) = 0 \text{ for all } (z_0, \lambda_0) \in S\}$.

Proposition 25. *Suppose $I = \langle g_1, \ldots, g_\kappa \rangle$, R, and N are ideals in $\mathbb{C}[z, \lambda]$ such that R radical and $I = R \cap N$. Then, for any $g \in \mathbf{I}(\mathbf{V}_\mathbb{C}(I))$ and any $(z_0, \lambda_0) \in \mathbb{C}^{p+1} \setminus \mathbf{V}_\mathbb{C}(N)$, there exist a neighborhood U of (z_0, λ_0) in $\mathbb{C}^{p+1}$ and rational functions $h_1, \ldots, h_\kappa$ on U such that $g = h_1 g_1 + \cdots + h_\kappa g_\kappa$ on U.*

Proof of Theorem 24. The Strong Hilbert Nullstellensatz and the hypothesis $\mathbf{V}_\mathbb{C}(\mathcal{I}) = \mathbf{V}_\mathbb{C}(\mathcal{I}_k)$ yield
$$\mathcal{I} \subset \sqrt{\mathcal{I}} = \mathbf{I}(\mathbf{V}_\mathbb{C}(\mathcal{I})) = \mathbf{I}(\mathbf{V}_\mathbb{C}(\mathcal{I}_k)).$$
From now on we complexify and assume $\Omega \subset \mathbb{C}$ and parameters $\lambda \in \mathbb{C}^p$ so that the averaged functions $f_j \in \mathbb{C}[z, \lambda]$ and, in particular we have that $f_j \in \mathbf{I}(\mathbf{V}_\mathbb{C}(\mathcal{I}))$ for any $j \in \mathbb{N}$. Let $\{f_{i_1}(z;\lambda), \ldots, f_{i_\kappa}(z;\lambda)\}$ be a minimal basis of $\mathcal{I}_k$. Hence for any f_j and any $(z_0, \lambda^*) \in \mathbb{C}^{p+1} \setminus \mathbf{V}_\mathbb{C}(\mathcal{N})$, by Proposition 25 there exists a neighborhood U of (z_0, λ^*) in $\mathbb{C}^{p+1}$ and κ rational functions $h_1, \ldots, h_\kappa$ such that, as analytic functions from U to $\mathbb{C}$, $f_j = h_1 f_{i_1} + \cdots + h_\kappa f_{i_\kappa}$ is valid on U for any $j \in \mathbb{N}$. This means that working with the germs at (z_0, λ^*) of the analytic functions involved, the displacement function d can be written, for (z, λ) in a neighborhood of (z_0, λ^*) and $|\varepsilon|$ sufficiently close to 0, as
$$d(z, \lambda, \varepsilon) = \sum_{j \geq 1} f_j(z;\lambda)\varepsilon^j = \sum_{q=1}^{\kappa} f_{i_q}(z;\lambda)[1 + \psi_q(z, \lambda, \varepsilon)]\varepsilon^{i_q}$$
where ψ_q are analytic functions with $\psi_q(z, \lambda, 0) = 0$. Then (see for example Proposition 6.1.2 of [12]) there are at most $\kappa - 1$ small (either positive or negative) zeros of $d(z, \lambda, .)$ for any (z, λ) sufficiently close to (z_0, λ^*). In other words, the number of (either positive or negative) isolated branches of the T-periodic solutions that can have equation (1), with parameters $\|\lambda - \lambda^*\| \ll 1$ and $|\varepsilon| \ll 1$, bifurcating from z_0 is at most $2(\kappa - 1)$. □

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